Three-Body Problem: Problems – IPhO 2011

Contestant: ChatGPT-4

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ChatGPT's Grade: For this problem, ChatGPT-4 got 4.4 out of 10. Not bad by IPhO standard. At this range of score, it might get an honorable mention award. Below is problem 1 out of 3 problems from IPhO year 2011.

1 A Three-body Problem and LISA

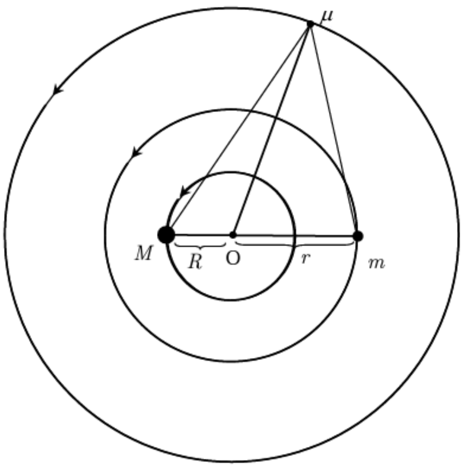

Figure 1 Coplanar orbits of three bodies.

1.1 Two gravitating masses M and m are moving in circular orbits of radii R and r, respectively, about their common centre of mass. Find the angular velocity ω_0 of the line joining M and m in terms of R, r, M, m and the universal gravitational constant G.

[1.5 points]

ChatGPT's Answer: To find the angular velocity ω_0 of the line joining M and m, we'll start by analyzing the forces acting on the system. The force between the two masses is given by Newton's law of gravitation:

$$F = G\frac{Mm}{(R+r)^2}$$

Since both masses are moving in circular orbits, we can equate the gravitational force with the centripetal force for each mass.

For mass M, the centripetal force is given by:

$$F_M = MR\omega_0^2$$

For mass m, the centripetal force is given by:

$$F_m = mr\omega_0^2$$

Now we can equate the gravitational force with the centripetal force for each mass:

$$G\frac{Mm}{(R+r)^2} = MR\omega_0^2$$

$$G\frac{Mm}{(R+r)^2} = mr\omega_0^2$$

Notice that we can divide the second equation by the first one to eliminate the gravitational constant G and the term $(R+r)^2$:

$$\frac{mr\omega_0^2}{MR\omega_0^2} = \frac{m}{M}$$

Now we can simplify the equation and solve for ω_0^2:

$$\frac{r}{R} = \frac{m}{M}$$

$$\omega_0^2 = \frac{G(M+m)}{(R+r)^3}$$

Finally, we can take the square root of both sides to get the angular velocity ω_0:

$$\omega_0 = \sqrt{\frac{G(M+m)}{(R+r)^3}}$$

ChatGPT's Grade: 1.5 out of 1.5

This is correct. All the reasoning steps and the equation solving are correct.

1.2 A third body of infinitesimal mass μ is placed in a coplanar circular orbit about the same centre of mass so that μ remains stationary relative to both M and m as shown in Figure 1 . Assume that the infinitesimal mass is not collinear with M and m. Find the values of the following parameters in terms of R and r :

[3.5 points]

1.1.1 distance from μ to M.

1.1.2 distance from μ to m.

1.1.3 distance from μ to the centre of mass.

ChatGPT's Answer: Since the infinitesimal mass μ remains stationary relative to both M and m, it must be located at one of the Lagrange points (specifically, L4 or L5). These points form equilateral triangles with the masses M and m.

Let's denote the distance from μ to M as d_M, the distance from μ to m as d_m, and the distance from μ to the center of mass as d_O.

1.1.1 Distance from μ to M:

Since $\triangle O\mu M$ is an equilateral triangle, we have:

$$d_M = R + r$$

1.1.2 Distance from μ to m:

Similarly, since $\triangle O\mu m$ is also an equilateral triangle, we have:

$$d_m = R + r$$

1.1.3 Distance from μ to the center of mass:

We can use the fact that the center of mass lies on the line segment connecting M and m, and the ratio of the distances from the center of mass to M and m is given by the inverse ratio of their masses:

$$\frac{d_O}{R} = \frac{m}{M} \Rightarrow d_O = \frac{m}{M}R$$

Now, we can use the Law of Cosines on the triangle $\triangle O\mu M$ to find d_O:

$$(R + r)^2 = R^2 + d_O^2 - 2Rd_O \cos 60°$$

Substitute the value of d_O from the equation above:

$$(R + r)^2 = R^2 + \left(\frac{m}{M}R\right)^2 - 2R\left(\frac{m}{M}R\right)\cos 60°$$

Simplify the equation and solve for d_O:

$$(R + r)^2 = R^2 + \frac{m^2}{M^2}R^2 - \frac{m}{M}R^2$$

$$(R + r)^2 = R^2\left(1 + \frac{m^2}{M^2} - \frac{m}{M}\right)$$

$$d_O = R\sqrt{1 + \frac{m^2}{M^2} - \frac{m}{M}}$$

So, the values of the parameters are:

1.1.1 $d_M = R + r$

1.1.2 $d_m = R + r$

1.1.3 $d_O = R\sqrt{1 + \frac{m^2}{M^2} - \frac{m}{M}}$

ChatGPT's Grade: 2/3.5

Very impressive reasoning that μ must form equilateral triangles with the masses M and m. A longer route is to start from the constraints that the non centripetal force due to the gravitation from both m and M cancels out. However, since the model does not give good enough reasons, I cannot give it good grades just yet. (0.5/1 for each of i and ii)

The final answer is almost correct except the minus terms should be plus. The law of cosines set up at the beginning caused it to be incorrect. (1/1.5 for iii)

[$^{B}_{A}$ Just to be sure that the model did not make things up and actually understand, I asked it to prove that the three points form an equilateral triangle. Below, the reasoning is still a bit circular since it's using the result to prove the result again, so it is not quite correct.]

ChatGPT's Answer: To prove that the Lagrange points L4 and L5 form equilateral triangles with masses M and m, let's consider the geometry of the problem and the forces acting on the infinitesimal mass μ.

Let's denote the angle $\angle MO\mu$ as θ. Since μ is stationary relative to both M and m, the net force acting on it should be zero. Thus, the gravitational forces acting on μ from M and m should be equal and opposite to the centripetal force required for its circular motion.

The gravitational force acting on μ from M is:

$$F_{M\mu} = G\frac{M\mu}{(R+r)^2}$$

The gravitational force acting on μ from m is:

$$F_{m\mu} = G\frac{m\mu}{d_m^2}$$

Since μ is stationary relative to M and m, it has the same angular velocity as them, which we found in the previous question:

$$\omega_0 = \sqrt{\frac{G(M+m)}{(R+r)^3}}$$

The centripetal force acting on μ is:

$$F_{c\mu} = \mu d_O \omega_0^2$$

Now let's consider the force equilibrium. The net force in the radial direction (along the line $O\mu$) should be zero:

$$F_{M\mu}\cos\theta - F_{m\mu}\cos\theta = 0$$

Since $\cos\theta \neq 0$, we can divide both sides by $\cos\theta$:

$$F_{M\mu} = F_{m\mu}$$

Substitute the expressions for the gravitational forces:

$$G\frac{M\mu}{(R+r)^2} = G\frac{m\mu}{d_m^2}$$

Since $\mu \neq 0$ and $G \neq 0$, we can cancel the terms:

$$\frac{M}{(R+r)^2} = \frac{m}{d_m^2}$$

Now, let's consider the force equilibrium in the tangential direction (perpendicular to the line $O\mu$):

$$F_{M\mu}\sin\theta + F_{m\mu}\sin\theta = F_{c\mu}$$

Since $\sin\theta \neq 0$, we can divide both sides by $\sin\theta$:

$$F_{M\mu} + F_{m\mu} = F_{c\mu}$$

Substitute the expressions for the forces:

$$G\frac{M\mu}{(R+r)^2} + G\frac{m\mu}{d_m^2} = \mu d_O \omega_0^2$$

Since $\mu \neq 0$, we can cancel the term:

$$G\frac{M}{(R+r)^2} + G\frac{m}{d_m^2} = d_O \omega_0^2$$

Now, substitute the expression for ω_0^2:

$$G\frac{M}{(R+r)^2} + G\frac{m}{d_m^2} = d_O \frac{G(M+m)}{(R+r)^3}$$

If μ is at the L4 or L5 point, it must satisfy the conditions derived above. For L4 and L5, the angle...

[$^{B}_{A}$ Note: the model got cut off due to long generation, probably. I asked the model to continue.]

Apologies for the abrupt ending. Let's continue the proof.

For L4 and L5, the angle θ is either 60° or 300°. Without loss of generality, let's assume $\theta = 60^\circ$. The other case will have a similar analysis.

Recall that we derived the following relation:

$$\frac{M}{(R+r)^2} = \frac{m}{d_m^2}$$

Now, let's use the Law of Cosines on $\triangle MO\mu$:

$$d_m^2 = R^2 + (R+r)^2 - 2R(R+r)\cos 60^\circ$$

$$d_m^2 = R^2 + (R+r)^2 - R(R+r)$$

Now, substitute the relation between M, m, R, r and d_m:

$$\frac{m(R+r)^2}{M} = R^2 + (R+r)^2 - R(R+r)$$

Multiply both sides by M:

$$m(R+r)^2 = M(R^2 + (R+r)^2 - R(R+r))$$

Now, substitute the expression for d_O in terms of the gravitational forces and angular velocity:

$$G\frac{M}{(R+r)^2} + G\frac{m}{d_m^2} = d_O\frac{G(M+m)}{(R+r)^3}$$

Since we know that $d_m = R + r$, we can simplify the equation:

$$G\frac{M}{(R+r)^2} + G\frac{m}{(R+r)^2} = d_O\frac{G(M+m)}{(R+r)^3}$$

Cancel the common terms:

$$M + m = d_O\frac{(M+m)}{(R+r)}$$

Now, divide both sides by $(M + m)$:

$$1 = d_O\frac{1}{(R+r)}$$

Finally, solve for d_O:

$$d_O = R + r$$

Since $d_O = R + r$, the triangle $\triangle MO\mu$ is an equilateral triangle with side length $R + r$. The same argument can be applied for the case when $\theta = 300°$. Thus, we have shown that the points L4 and L5 form equilateral triangles with masses M and m.

1.3 Consider the case $M = m$. If μ is now given a small radial perturbation (along Oμ), what is the angular frequency of oscillation of μ about the unperturbed position in terms of ω_0? Assume that the angular momentum of μ is conserved. [3.2 points]

ChatGPT's Answer: To find the angular frequency of oscillation of μ about the unperturbed position after a small radial perturbation, we will consider the conservation of angular momentum and the effective potential energy of μ in the system.

Let L_μ be the angular momentum of μ. Since it is conserved, we have:

$$L_\mu = \mu d_O \omega_0$$

[$^{B}_{A}$ this is not quite correct, since it should be d_O^2 instead of d_O for the angular momentum term]

Now, let's denote the radial perturbation by Δr, and the new distance from the center of mass O to the perturbed position of μ as $d'_O = d_O + \Delta r$. Since the angular momentum is conserved, we can write:

$$L_\mu = \mu d'_O \omega'$$

We can now equate the two expressions for L_μ:

$$\mu d_O \omega_0 = \mu d'_O \omega'$$

Since $\mu \neq 0$, we can cancel the term and solve for ω':

$$\omega' = \frac{d_O}{d'_O}\omega_0$$

Now, let's consider the effective potential energy U_{eff} of μ. It consists of the gravitational potential energy due to the masses M and m, and the centrifugal potential energy due to the circular motion of μ:

$$U_{\text{eff}} = -G\frac{M\mu}{d_{M\mu}} - G\frac{m\mu}{d_{m\mu}} + \frac{1}{2}\mu d'^2_O \omega'^2$$

[$^{B}_{A}$ this term is missing the energy involving the oscillation along the radial direction due to perturbation]

For small radial perturbations Δr, we can approximate $d_{M\mu} \approx d_{m\mu} \approx d'_O$, and we can expand U_{eff} in a Taylor series around d_O. Since we are considering the case $M = m$, we have:

$$U_{\text{eff}} \approx -2G\frac{M\mu}{d_O} + \frac{1}{2}\mu d_O^2 \omega_0^2 + \mu d_O \Delta r \omega_0^2$$

Now, we can differentiate U_{eff} with respect to Δr and set the derivative to zero to find the equilibrium position:

$$\frac{\partial U_{\text{eff}}}{\partial(\Delta r)} = \mu d_O \omega_0^2 = 0$$

Since $\mu \neq 0$, we have:

$$\omega_0^2 = 0$$

This result implies that there is no oscillation in the radial direction for the case $M = m$. The system is neutrally stable, meaning that any small radial perturbation will not cause oscillations of μ about the unperturbed position.

ChatGPT's Grade: 0.5/3.2

The final answer is numerically incorrect, and also principally incorrect (no oscillation was the conclusion). Most of the derivation logic is actually correct. However, there are several key places that ChatGPT-4 misses. First, the angular momentum is proportional to d_O^2. I would have expected it to master this concept, so it is a bit surprising. The other miss is the expression of energy – it is missing a key part involving the radial oscillation, which makes the final result not making sense.

Even if the model sets up the energy expression correctly, it does not yet demonstrate how to derive the radial oscillation frequency. This can be done by finding a way to write the acceleration of the perturbed term $\xi = \Delta r$ as $\xi'' = -\omega^2\xi$. Then, ω will be the answer, expressed in terms of other variables. This step would involve differentiating the energy with respect to **time** rather than with respect to Δr like the answer above (although that might have worked, but it might be more complicated). After that, it would involve a series of first order approximation via Taylor expansions. The logic is simple but keeping track of all the details might be tricky. After showing that $\xi'' = -\omega^2\xi$ without any extra terms (which should have all canceled out), this would imply that there is actually a radial oscillation. If there were other terms such as a constant, it would otherwise imply a different motion.

The final answer should be $\omega_{\text{oscillation}} = \frac{\sqrt{7}}{2}\omega_0$

The Laser Interferometry Space Antenna (LISA) is a group of three identical spacecrafts for detecting low frequency gravitational waves. Each of the spacecrafts is placed at the corners of an equilateral triangle as shown in Figure 2 and Figure 3. The sides (or 'arms') are about 5.0 million kilometres long. The LISA constellation is in an earth-like orbit around the Sun trailing the Earth by 20°. Each of them moves on a slightly inclined individual orbit around the Sun. Effectively, the three spacecrafts appear to roll about their common centre one revolution per year.

They are continuously transmitting and receiving laser signals between each other. Overall, they detect the gravitational waves by measuring tiny changes in the arm lengths using interferometric means. A collision of massive objects, such as blackholes, in nearby galaxies is an example of the sources of gravitational waves.

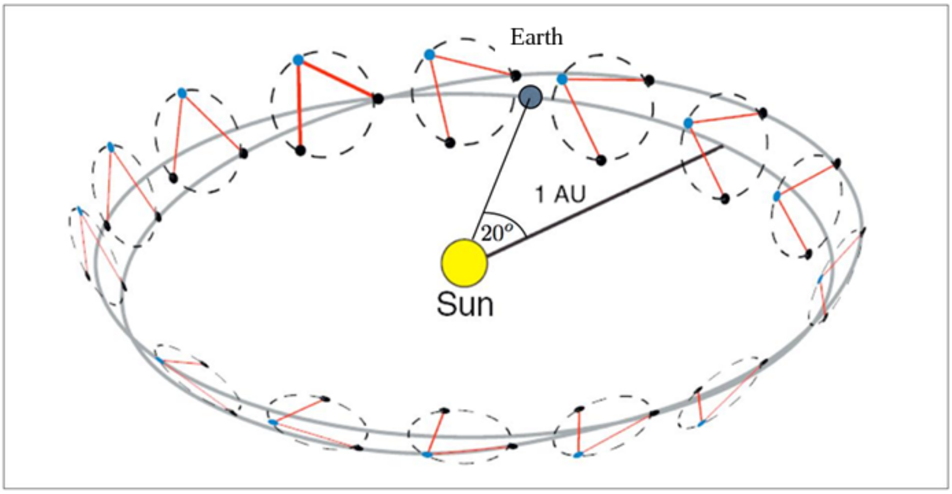

Figure 2 Illustration of the LISA orbit. The three spacecraft roll about their centre of mass with a period of 1 year. Initially, they trail the Earth by 20°. (Picture from D.A. Shaddock, "An Overview of the Laser Interferometer Space Antenna", *Publications of the Astronomical Society of Australia*, 2009, **26**, pp.128-132.).

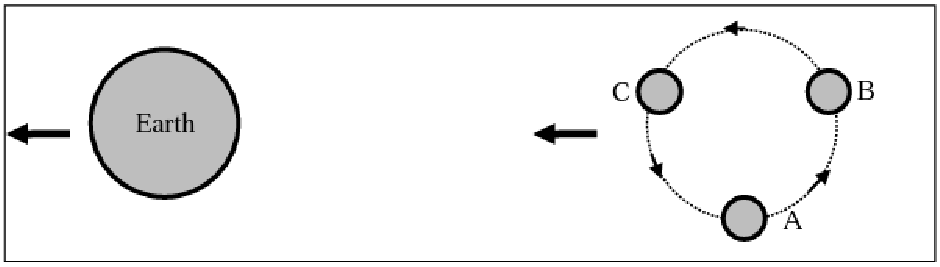

Figure 3 Enlarged view of the three spacecrafts trailing the Earth. A, B and C are the three spacecrafts at the corners of the equilateral triangle.

1.4 In the plane containing the three spacecrafts, what is the relative speed of one spacecraft with respect to another?

[1.8 point]

ChatGPT's Answer: To find the relative speed of one spacecraft with respect to another, we need to consider the motion of the spacecrafts around the Sun and their motion around the common center of mass.

Let's denote the radius of the Earth-like orbit around the Sun as R_{orbit}, and the length of each arm of the equilateral triangle as L_{arm}. Given that the LISA constellation revolves around the Sun once per year, we can find the angular velocity of the spacecrafts around the Sun, ω_{orbit}, as:

$$\omega_{\text{orbit}} = \frac{2\pi \text{ rad}}{1 \text{ year}}$$

Now, let's find the linear speed of each spacecraft in their orbit around the Sun, v_{orbit}:

$$v_{\text{orbit}} = R_{\text{orbit}} \cdot \omega_{\text{orbit}}$$

Next, we need to consider the motion of the spacecrafts around their common center of mass. Since the LISA constellation rolls about the common center with a period of 1 year, the angular velocity of the spacecrafts around the common center, ω_{center}, is the same as their angular velocity around the Sun:

$$\omega_{\text{center}} = \omega_{\text{orbit}}$$

$[^{\text{B}}_{\text{A}}$ correct$]$

Now, let's find the linear speed of each spacecraft around the common center, v_{center}:

$$v_{\text{center}} = \frac{L_{\text{arm}}}{2} \cdot \omega_{\text{center}}$$

$[^{\text{B}}_{\text{A}}$ this is almost correct, however, the radius is actually $\frac{L}{\sqrt{3}}$, not $\frac{L}{2}]$

Here, we used $\frac{L_{\text{arm}}}{2}$ as the distance from the center of mass to the midpoint of the side, which is the effective radius of the circular motion of each spacecraft around the common center.

Now, we need to find the relative speed of one spacecraft with respect to another. The motion of the spacecrafts around the Sun and around the common center are perpendicular to each other. Therefore, we can use the Pythagorean theorem to find the magnitude of the relative speed, v_{relative}:

$$v_{\text{relative}} = \sqrt{v_{\text{orbit}}^2 + v_{\text{center}}^2}$$

$[^{\text{B}}_{\text{A}}$ not exactly the correct logic to get relative speed of one spacecraft from another$]$

Substitute the expressions for v_{orbit} and v_{center}:

$$v_{\text{relative}} = \sqrt{(R_{\text{orbit}} \cdot \omega_{\text{orbit}})^2 + \left(\frac{L_{\text{arm}}}{2} \cdot \omega_{\text{center}}\right)^2}$$

Since $\omega_{\text{orbit}} = \omega_{\text{center}}$, we can simplify the expression:

$$v_{\text{relative}} = \omega_{\text{orbit}} \sqrt{R_{\text{orbit}}^2 + \left(\frac{L_{\text{arm}}}{2}\right)^2}$$

Given the values for the Earth-like orbit radius, $R_{\text{orbit}} \approx 1$ AU (astronomical unit), and the arm length, $L_{\text{arm}} = 5.0 \times 10^6$ km, we can plug these values and calculate the relative speed. However, the answer will be in terms of the angular velocity ω_{orbit}, which is in units of rad/year. To convert the answer to a more common unit like km/s, we need to use appropriate conversion factors for the distances and time. [$^{B}_{A}$ Due to the incorrect logic setup for the relative speed, the final answer is incorrect.]

ChatGPT's Grade: 0.4/1.8

This problem is actually quite simple, but the long problem description may be confusing.

The model obtains the speed around the center almost correctly. The incorrect part is the radius around O.

Since the magnitude of relative velocity of one spacecraft to another is the same, we can consider a simple setup in a Cartesian coordinate and express the velocity of each spacecraft in terms of $\hat{i}, \hat{j}$. It is simple to derive that the magnitude of such relative velocity is $\sqrt{3}v$ where v is the speed around the center, which is $\omega \frac{L}{\sqrt{3}}$.

The final answer is 966 m/s.