ON RECURRENCE AND POWER WEAKLY MIXING PROPERTIES OF INFINITE MEASURE PRESERVING TRANSFORMATIONS

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Abstract. We explore two properties of infinite measure preserving transformations. We examine a T that is recurrent but not 2-recurrent which implies that $T \times T$ is not conservative. We have added a proof that $T \times T^2$ is also not conservative.

In addition, we introduce conditions that imply power weak mixing property for cutting and stacking transformations with arbitrary number of cuts and tower spacers.

1. Preliminary

In this paper, we are interested in a certain class of transformations defined recursively on $(X, \mathfrak{L}(X), \mu)$ for $X \subset \mathbb{R}$ called **cutting and stacking** transformations. It can be shown that cutting and stacking transformations are invertible and measure-preserving. Let $X \subset \mathbb{R}$ and $(X, \mathfrak{L}(X), \mu)$ be a Lebesgue measure space on X. A general construction of cutting and stacking transformations can be represented by a *column*, an ordered list consisting of equal-length intervals. We always start from column C_0 which consists on one interval $I_{0,0}$. Generally, we can write the nth column, C_n, as

$$C_n = \{I_{n,0}, I_{n,1}, \cdots, I_{n,h_n-1}\}$$

where each $I_{n,j}$ is called the j^{th} level in the column C_n and h_n is the height or the number of levels in C_n.

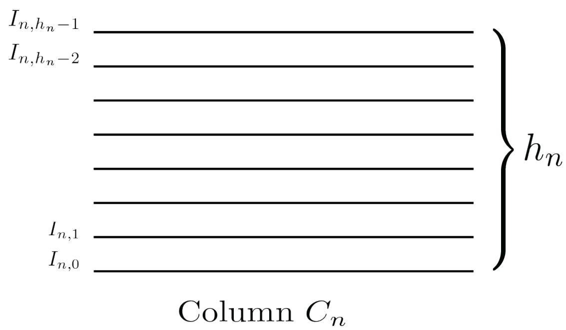

FIGURE 1. Column C_n

The transformation is partially defined in each C_n by the implicit order in that T map points (except in the top level) to the points directly above it in the column structure. Note that $T(I_{n,j}) = I_{n,j+1}$ for $j < h_n - 1$. The mapping on the topmost level I_{n,h_n-1} will be defined further in higher columns.

A construction of C_{n+1} from C_n and spacers called cutting and stacking completely specify a transformation. The column C_{n+1} is formed by cutting C_n vertically into r_n **subcolumns** of equal length. Spacers of length equal to that of a subcolumn can be added on top.

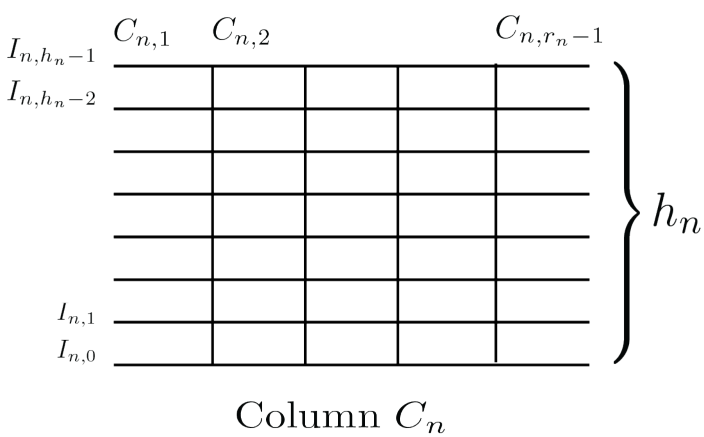

FIGURE 2. Subcolumns in C_n

Spacers are added in such a way the transformation are defined on all points in the domain X. These subcolumns of C_n are stacked from left to right to form C_{n+1}. That is, the top level of one subcolumn is mapped to the bottom level of the subcolumn to its right. Observe that the column C_{n+1} preserves the transformation defined in C_n.

The cutting and stacking transformations are part of a more general class of transformations called *rank one*. We give the definition below.

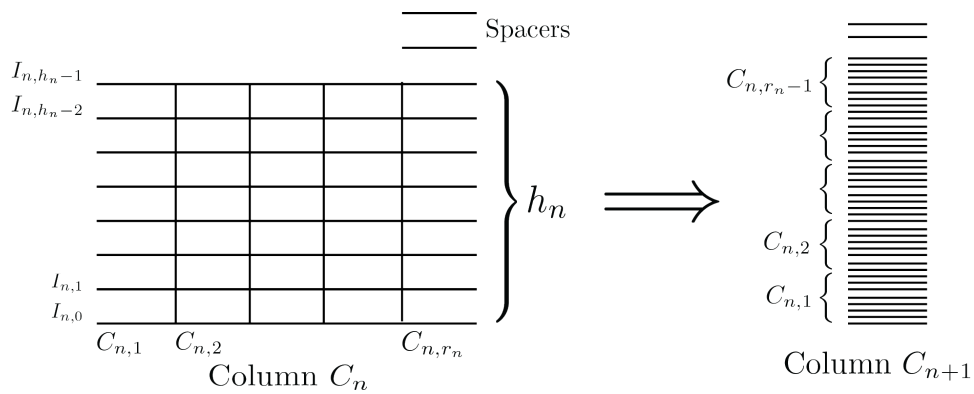

FIGURE 3. Cutting and stacking

Definition 1.1. A transformation is of **rank one** if it can be completely described by a sequence of columns $\{C_n\}_{n\geq 0}$ with the following properties.

(1) C_{n+1} is a refinement of C_n. That is, a given level in C_n can be written as a finite union of levels in C_{n+1}.

(2) The collection of $\mathcal{C} \equiv \bigcup_{n=0}^{\infty} C_n$ forms a sufficient semi-ring. That is, for any given measurable set A of positive measure, for a given $\epsilon > 0$, there exists elements H that is a finite union of elements in $\mathcal{C}$ such that
$$\mu(A \triangle H) < \epsilon.$$

(3) In addition, the union of elements in C_n exhausts the space mod μ. That is,
$$\mu\left[X \setminus \left(\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{J_n} I_{n,j}\right)\right] \to 0.$$

2. Non 2-Recurrent Transformation

Definition 2.1. Let $(X, \mathcal{S}, \mu)$ be a measure space. A transformation T is said to be **recurrent** if for a given set A of positive measure, there exists a null set $N \subset A$ such that for any $x \in A \backslash N$, there exists an integer $n = n(x, A) > 0$ such that $T^n(x) \in A$.

That is, a transformation is recurrent if almost every point in a set A eventually evolve under the transformation back to the original set. We note that a recurrent transformation is also equivalent to a conservative transformation.

Definition 2.2. Let $(X, \mathcal{S}, \mu)$ be a measure space. A transformation T is **conservative** if for a given set A of positive measure, there exists an integer $n = n(A) > 0$ such that

$$\mu(T^{-n}(A) \cap A) > 0.$$

Theorem 2.3. A measure preserving transformation T is recurrent if and only if it is conservative.

Readers may refer to proof in [9].

A transformation T is said to be k recurrent $(k > 0)$ if for a given set A of positive measure, there exists an integer $n > 0$ such that

$$\mu(A \cap T^{-n}(A) \cap T^{-2n}(A) \ldots \cap T^{-kn}(A)) > 0.$$

Theorem 2.4. *(Furstenberg Multiply Recurrence Theorem)* A measure preserving transformation T on a measure space $(X, \mathcal{S}, \mu)$ is multiply recurrent if for any integer $k > 0$, T is k recurrent.

It turns out that despite the seemingly strong restriction, any measure preserving transformation on a *finite Lebesgue measure* is multiply recurrent. This is one of the major theorems in ergodic theory. Readers may refer to the full proof in [7].

In this chapter, we will provide an example of an *infinite* measure preserving transformation that is conservative(recurrent) but not 2-recurrent and therefore is not multiply recurrent. This example demonstrates the difference between finite and infinite measure space in that a transformation T in a finite Lebesgue measure space need not be multiply recurrent.

We note that in the literature, examples of infinite measure-preserving transformations that are not multiply recurrent are shown in [6], [1], and [4].

2.1. **Construction.**

We show that the cutting and stacking transformation T defined in [2] is recurrent but not 2-recurrent. Below details the construction of transformation.

Let the first column C_1 be $[0, 1)$. We obtain column C_{n+1} by cutting column C_n into r_n subcolumns. Then, place $(2^{r_n - i} - 1)h_n$ spacers on top of the

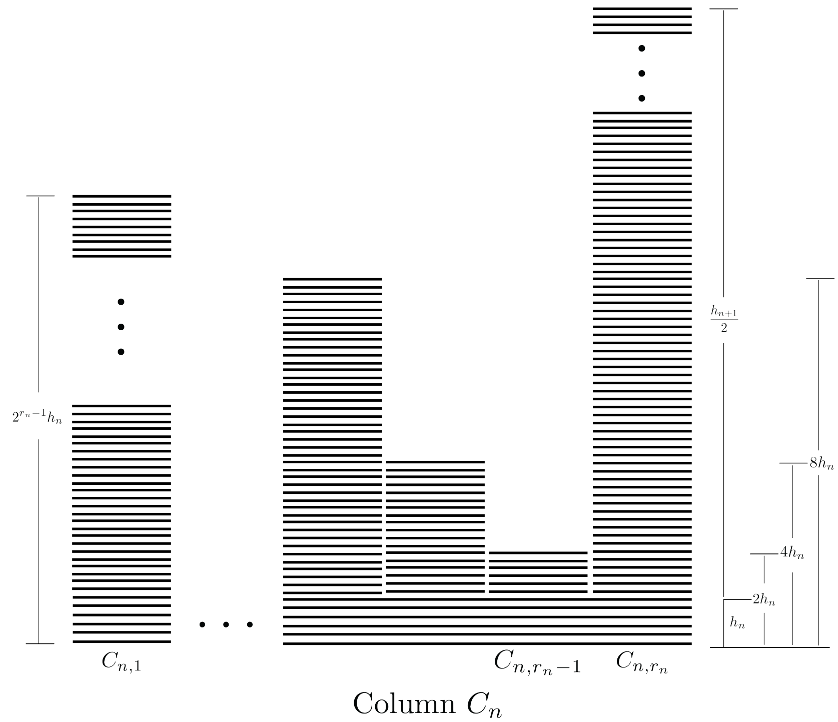

FIGURE 4. Construction of T.

i^{th} subcolumn, $i < r_n$. For the last subcolumn, place $r_nh_n + \sum_{i=1}^{r_n-1}(2^{r_n-i} - 1)h_n = h_{n+1}/2$ spacers. Then, stack the subcolumns with spacers from left to right. Figure 2.1 describes the construction of T. Note that the number of levels below the last column spacers is $r_nh_n + \sum_{i=1}^{r_n-1}(2^{r_n-i} - 1)h_n$, which is exactly the height of the spacers.

2.2. **Recurrence Property of T.** We will give a proof that if r_n is bounded, then T is a conservative transformation.

Theorem 2.5. Suppose there exists a positive integer M such that $r_n < M$, then the transformation T defined by the sequence r_n is conservative.

Proof. Given a measurable set A, since levels form a sufficient semi ring, there exists a level I in some column C_n such that it is $1 - \frac{1}{2M}$ full of A.

Then, for each subinterval $I^{[k]}, k \in \{1, \ldots, r_n\}$,

$$\begin{aligned}
\mu(I^{[k]} \cap A) &= \mu(I \cap A) - \mu(\bigsqcup_{j \notin k} I^{[j]} \cap A) \\
&> (1 - \frac{1}{2M})\mu(I) - \mu(\bigsqcup_{j \notin k} I^{[j]}) \\
&= (1 - \frac{1}{2M}) r_n \mu(I^{[k]}) - (r_n - 1)\mu(I^{[k]}) \\
&= (1 - \frac{r_n}{2M})\mu(I^{[k]}) \\
&\geq \frac{1}{2}\mu(I^{[k]})
\end{aligned}$$

That is, any subinterval is $\frac{1}{2}$ full A. Since the distance between the $\mu(I^{[r_n - 1]})$ and $\mu(I^{[r_n]})$ is $2h_n$, we have that

$$\mu(T^{2h_n} I \cap I) \geq \mu(I^{[k]})$$

Thus, letting $m = 2h_n$,

$$\begin{aligned}
\mu(T^m A \cap A) &\geq \mu(T^m I \cap I) - 2\mu(I \backslash A) \\
&> \mu(I^{[k]}) - 2 \cdot \frac{1}{2}\mu(I^{[k]}) = 0.
\end{aligned}$$

Therefore, the transformation T is conservative. □

Theorem 2.6. The transformation T is not 2-recurrent.

Proof. Note that the transformation is invertible and measure-preserving; therefore, it is equivalent to consider the forward images instead of the preimages in the definition of k-recurrence. Let A be the top level of a column C_n. We will show that for any integer $m > 0$, $A \cap T^m(A) \cap T^{2m}(A) = \emptyset$.

Keeping the same notation, let r_n be the number of subcolumns of C_n. Let $A_{n,i}$ where $i = 1, \ldots, r_n$ denote the copy of A in the i^{th} subcolumn $C_{n,i}$. First, we consider the case where $m \leq h_{n+1}/2$. Note that in this scenario, $T^m(A)$ is still in the column C_n as the height of the last subcolumn C_{n,r_n} is $h_{n+1}/2$. Therefore, A can overlap with $T^m(A)$ only when

$$A_{n,j} \cap T^m A_{n,i} \cap T^{2m} A_{n,q} \neq \emptyset \tag{4.4.3.1}$$

for some $q, i, j \in \{1, \ldots, r_n\}$. It is clear that by our notation, $q < i < j$.

We suppose for contradiction that there exists integers q, i, j such that equation (4.4.3.1) is true. Note that the distance between the $A_{n,i}$ and $A_{n,j}$, assuming $i < j$, can be shown to be

$$2^{r_n - j} \cdot \left(\sum_{p=1}^{j-i} 2^p \right) \cdot h_n \equiv d^n_{i,j}. \tag{4.4.3.2}$$

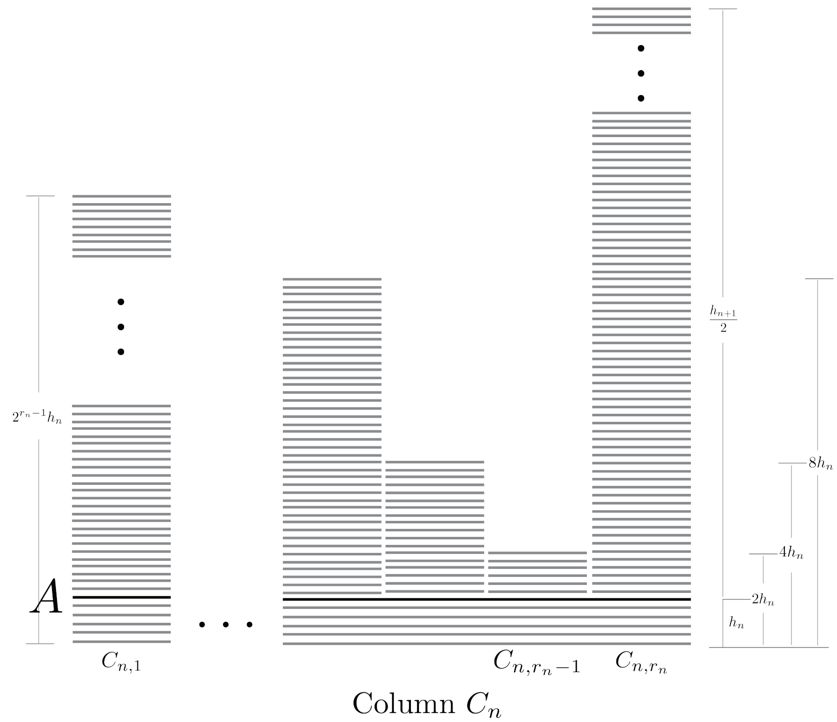

FIGURE 5. Top level A of column C_n.

From the supposition that $A_{n,j}$ and $T^m(A_{n,i})$ intersect, we have that $m = d^n_{j,i}$ so that $T^m(A_{n,i}) = A_{n,j}$. From $A_{n,j} \cap T^{2m}(A_{n,q}) \neq \emptyset$, we also have $2m = d^n_{j,q}$. Both overlaps are possible only when $2d^n_{j,i} = d^n_{j,q}$. That is,

$$2 \cdot d^n_{j,i} = d^n_{j,q}$$

$$2 \cdot 2^{r_n - j} \cdot \left(\sum_{p=1}^{j-i} 2^p \right) \cdot h_n = \cdot 2^{r_n-j} \cdot \left(\sum_{p=1}^{j-q} 2^p \right) \cdot h_n$$

$$2 \cdot \left(\sum_{p=1}^{j-q} 2^p + \sum_{p=j-q+1}^{j-i} 2^p \right) = \left(\sum_{p=1}^{j-q} 2^p \right)$$

$$\left(\sum_{p=1}^{j-q} 2^p \right) + 2 \cdot \left(\sum_{p=j-q+1}^{j-i} 2^p \right) = 0 \tag{4.4.3.3}$$

which is a contradiction as all the terms on the left are nonzero. We conclude that for $m \leq h_{n+1}/2$,

$$A \cap T^m(A) \cap T^{2m}(A) = \emptyset \tag{4.4.3.4}$$

Next, consider the case where $h_{n+1}/2 < m \leq h_{n+2}/2$. In column C_{n+1}, observe that this distance between the top level of copies of A and the

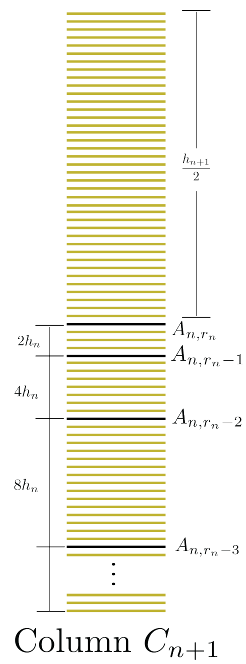

FIGURE 6. Copies of A in column C_{n+1}.

bottom level of copies of A is exactly $h_{n+1}/2 - h_n$. We will only need that it is bounded by $h_{n+1}/2$. In C_{n+2}, the copies of A are bands, whose width are bounded above by $h_{n+1}/2$.

Top levels are separated from top to bottom by distance $2h_{n+2}, 4h_{n+2}$, and so on. (as is similar to previously in column C_{n+1}.) Let $A_{n+1,j}$, $A_{n+1,i}$, $A_{n+1,q}$ be the unions of copies of A, where $j, i, q \in \{1, \ldots, r_{n+1}\}$ and $j > i > q$. We call this structure a band, which implies that a copy of A in the given subcolumn has more than one level. The distance between the top level of band $A_{n+1,j}$ and band $A_{n+1,i}$ is

$$d_{j,i}^{n+1} = 2^{r_{n+1}-j} \cdot \left(2 + 4 + \ldots + 2^{j-i}\right) \cdot h_{n+1} \tag{4.4.3.5}$$

The condition of m in order to have $T^m(I)$ overlaps A is

$$m \in \left(d_{j,i}^{n+1} - h_{n+1}/2, d_{j,i}^{n+1} + h_{n+1}/2\right). \tag{4.4.3.6}$$

Clearly,

$$2m \in \left(2d_{j,i}^{n+1} - h_{n+1}, 2d_{j,i}^{n+1} + h_{n+1}\right) \equiv R_2 \tag{4.4.3.7}$$

If it is the case that $A \cap T^m(A) \cap T^{2m}(A) \neq \emptyset$, the range of $2m$ needs to overlap with the range $R_1 \equiv \left(d_{j,q}^{n+1} - h_{n+1}/2, d_{j,q}^{n+1} + h_{n+1}/2\right)$. Then, the relationship between $d_{j,q}^{n+1}$ and $d_{j,i}^{n+1}$ is

$$\begin{aligned}
d_{j,q}^{n+1} &= 2^{r_n - j} \cdot h_{n+1} \cdot \left(\sum_{p=1}^{j-q} 2^p \right) \\
&= 2^{r_n - j} \cdot h_{n+1} \cdot 2 \cdot \left(\sum_{p=0}^{j-q-1} 2^p \right) \\
&= 2^{r_n - j} \cdot h_{n+1} \cdot 2 \cdot \left(1 + \sum_{p=1}^{j-q-1} 2^p \right) \\
&= 2^{r_n - j} \cdot h_{n+1} \cdot 2 + 2^{r_n - j} \cdot h_{n+1} \cdot 2 \cdot \left(\sum_{p=1}^{j-q-1} 2^p \right) \\
&= 2^{r_n - j} \cdot 2h_{n+1} + 2d_{j,q+1}^{n+1} \\
&\geq 2h_{n+1} + 2d_{j,i}^{n+1} \qquad (4.4.3.8)
\end{aligned}$$

We may write $d_{j,q}^{n+1}$ as $d_{j,q}^{n+1} = 2h_{n+1} + 2d_{j,i}^{n+1} + \delta$ where δ is nonnegative by equation 4.4.3.8.

Then,

$$\begin{aligned}
R_1 &= \left(d_{j,q}^{n+1} - h_{n+1}/2, d_{j,q}^{n+1} + h_{n+1}/2 \right) \\
&= \left(2d_{j,i}^{n+1} + 2h_{n+1} + \delta - h_{n+1}/2, 2d_{j,i}^{n+1} + 2h_{n+1} + \delta + h_{n+1}/2 \right) \\
&= \left(2d_{j,i}^{n+1} + \frac{3}{2}h_{n+1} + \delta, 2d_{J,Q}^{n+1} + \frac{5}{2}h_{n+1} + \delta \right) \qquad (4.4.3.7)
\end{aligned}$$

It is clear that R_1 does not overlap with R_2. We conclude that for $h_{n+1}/2 < m \leq h_{n+2}/2$,

$$A \cap T^m(A) \cap T^{2m}(A) = \emptyset.$$

One can use induction to show that for any integer $k > 0$, for an integer m such that $h_{n+k}/2 < m < h_{n+k+1}/2$, $A \cap T^m(A) \cap T^{2m}(A) = \emptyset$. The argument is similar to what is presented for the case $k = 1$.

□

2.3. **Cartesian Product of T.** It has been shown in [2] that if the growth rate of r_n is sufficiently large, $T \times T$ is not a conservative transformation. We present a detailed proof below.

Theorem 2.7. If $\sum_{n=1}^{\infty} \frac{1}{r_n} < \infty$, then $T \times T$ is not conservative.

Proof. Suppose $\{r_k\}$ is a sequence of positive integers with each $r_k \geq 2$ such that $\sum_{k=1}^{\infty} \frac{1}{r_k}$ is finite. Then, there exists an integer $n > 0$ such that

$$\sum_{k=n}^{\infty} \frac{1}{r_k} < 1.$$

Let A be the top level of C_n. For $k \geq n$, let $C_{k,l}$ denote the l^{th} subcolumn of column C_k. Let $A_{k,l} = A \cap C_{k,l}$ denote copies of A in subcolumn $C_{k,l}$. Let $\nu = \mu \times \mu$ be the product measure. For any $l \leq r_k$, the measure of a rectangle $A_{k,l} \times A_{k,l}$ is

$$\nu\left(A_{k,l} \times A_{k,l}\right) = \mu(A_{k,l}) \cdot \mu(A_{k,l}) = \frac{1}{r_k}\mu(A) \cdot \frac{1}{r_k}\mu(A).$$

Thus,

$$\nu\left(\bigcup_{l=1}^{r_k}(A_{k,l} \times A_{k,l})\right) = r_k \cdot \nu(A_{k,l} \times A_{k,l}) = \frac{1}{r_k}\mu(A)^2.$$

Define

$$E = (A \times A) \setminus \left(\bigcup_{k=n}^{\infty}\bigcup_{l=1}^{r_k}(A_{k,l} \times A_{k,l})\right).$$

Note that the measure of E is

$$\nu(E) = \nu(A \times A) - \nu\left(\bigcup_{k=n}^{\infty}\bigcup_{l=1}^{r_k}(A_{k,l} \times A_{k,l})\right) \geq \mu(A)^2(1 - \sum_{k=n}^{\infty}\frac{1}{r_k}) > 0.$$

We will show by induction that for any integer $k \geq n$,

$$\nu((T \times T)^i E \cap E) = 0 \tag{1}$$

for $1 \leq i \leq h_k$.

For the base case ($k = n$), since A is a single level with C_n with spacers of height at least h_n and $E \subset A \times A$, then

$$\nu[(T \times T)^i E \cap E] \leq \nu[(T \times T)^i A \times A \cap (A \times A)] = 0$$

for all $1 \leq i \leq h_n$.

Next, suppose that $\nu((T \times T)^i E \cap E) = 0$ is true for $1 \leq i \leq h_k$, $k > n$. We will show that (1) is true for $h_k < i \leq h_{k+1}$.

Let l be an integer and $0 \leq l < r_k$. Since we place spacers of height $(2^{r_k - l} - 1)h_k$ on subcolumn $C_{k,l}$. We can see that for $h_k < i \leq (2^{r_k-l}-1)h_k$, $\mu(T^i(A_{k,l})) \cap A_k = 0$.

In addition, the number of iterations from the bottom level of subcolumn $C_{k,l}$ to reach the first spacer on the last subcolumn (C_{k,r_k}) is

$$\begin{aligned}
\sum_{i=l}^{r_k} h_k + \sum_{i=l}^{r_k-1}(2^{r_k-i} - 1)h_k &= (r_k - l + 1)h_k + \sum_{i=l}^{r_k-1} 2^{r_k-i}h_k - (r_k - 1 - l + 1)h_k \\
&= h_k + h_k \sum_{i=l}^{r_k-1} 2^{r_k-i} \\
&= h_k + h_k\left(\frac{2^{r_k-l}}{1/2} - \frac{2^{r_k-r_k}}{1/2}\right) \\
&= h_k + 2^{r_k-l+1}h_k - 2h_k \\
&= (2^{r_k-l+1} - 1)h_k.
\end{aligned}$$

Thus, $\mu(T^i C_{k,l} \cap C_k) = 0$ for $(2^{r_k - l+1} - 1)h_k \leq i \leq H - h_k$ where H the number of spacers in the last subcolumn. We can extend this result up to $i \leq h_{k+1}$ as follow.

Consider the column C_{k+1}. For $(2^{r_k-l+1} - 1)h_k \leq i \leq H - h_k$, we have that $T^i(C_{k,l})$ is contained in the spacers on the subcolumn C_{k,r_k}. In C_{k+1}, the spacers become full levels, with additional spacers for column C_{k+1}, with height greater or equal to h_{k+1} on top. Therefore, if $(2^{r_k-l+1} - 1)h_k \leq i \leq h_{k+1}$, $T^i(C_{k,l})$ is either in the spacers of C_k or in those of C_{k+1}. This implies

$$\mu(T^i C_{k,l} \cap C_k) = 0$$

for all i such that $(2^{r_k-l+1} - 1)h_k \leq i \leq h_{k+1}$. Also, since $A_{k,l} \subset C_{k,l}$ and $A_k \subset C_k$,

$$\mu[T^i A_{k,l} \cap A_k] = 0.$$

We conclude that if $\mu(T^i C_{k,l} \cap C_k) > 0$, then $i \leq (2^{r_k-l} - 1)h_k$ and $i \geq (2^{r_k-l+1} - 1)h_k$. Define $I_l = \{h_k < i \leq h_{k+1} : \mu(T^i C_{k,l} \cap C_k) > 0\}$. Thus,

$$I_l \subset ((2^{r_k-l} - 1)h_k, (2^{r_k-l+1} - 1)h_k).$$

Denote $((2^{r_k-l} - 1)h_k, (2^{r_k-l+1} - 1)h_k)$ as J_l. Let m be a positive integer distinct from l. Assume without loss of generality that $m > l$. We can see that

$$(2^{r_k-m+1} - 1)h_k \leq (2^{r_k-l} - 1)h_k.$$

Thus, the intervals J_l and J_m do not intersect which implies $I_l \cap I_m = \emptyset$ for $l \neq m$. Observe that

$$\begin{aligned}
\nu[(T \times T)^i(C_{k,l} \times C_{k,m}) \cap (C_k \times C_k)] &= \nu[T^i C_{k,l} \times T^i C_{k,m} \cap C_k \times C_k] \\
&= \nu[(T^i C_{k,l} \cap C_k) \times (T^i C_{k,m} \cap C_k)] \\
&= \mu[T^i C_{k,l} \cap C_k] \cdot \mu[T^i C_{k,m} \cap C_k].
\end{aligned}$$

Since $I_l \cap I_m = \emptyset$, if $\mu[T^i C_{k,l} \cap C_k] \neq 0$, then $\mu[T^i C_{k,m} \cap C_k] = 0$. Thus,

$$\nu[(T \times T)^i(C_{k,l} \times C_{k,m}) \cap (C_k \times C_k)] = 0.$$

Since $A_{k,i} \subset C_{k,i}$ and $A_k \subset C_k$, it follows that

$$\nu[(T \times T)^i(A_{k,l} \times A_{k,m}) \cap (A \times A)] = 0. \tag{2}$$

Partition E into two parts as follow.

$$E = \left(E \cap \bigcup_{l \neq m} A_{k,l} \times A_{k,m} \right) \sqcup \left(E \cap \bigcup_{l=1}^{r_k} A_{k,l} \times A_{k,l} \right) \equiv E' \sqcup E''$$

Then,

$$\nu[(T\times T)^i E'\cap E] = \nu\left[\left((T\times T)^i\left(\bigcup_{l\neq m} A_{k,l}\times A_{k,m}\cap E\right)\right)\cap E\right]$$

$$\leq \nu\left[\left((T\times T)^i\left(\bigcup_{l\neq m} A_{k,l}\times A_{k,m}\right)\right)\cap E\right]$$

$$(E\subset A\times A) \qquad \leq \nu\left[\left(\bigcup_{l\neq m}\big((T\times T)^i A_{k,l}\times A_{k,m}\big)\right)\cap (A\times A)\right]$$

$$\leq \sum_{l\neq m}\nu\left[\big((T\times T)^i A_{k,l}\times A_{k,m}\big)\cap (A\times A)\right]$$

$$(\text{From (2)}) \qquad = 0.$$

Next, to show that $\nu[(T\times T)^i E''\cap E]=0$, consider $\nu[E\cap(A_{k,l}\times A_{k,l})]$. From definition, $E = A\times A\backslash[\bigcup_{k=n}^{\infty}\bigcup_{l=1}^{r_k}(A_{k,l}\times A_{k,l})]$. This directly implies that $E\cap A_{k,l}\times A_{k,l}=\emptyset$ and thus

$$\nu[E\cap A_{k,l}\times A_{k,l}]=0.$$

Then,

$$\nu[(T\times T)^i E''\cap E] = \nu\left[(T\times T)^i\left(E\cap\bigcup_{l=1}^{r_k}A_{k,l}\times A_{k,l}\right)\cap E\right]$$

$$\leq \sum_{j=1}^{r_k}\nu\left[(T\times T)^i\left(E\cap (A_{k,l}\times A_{k,l})\right)\cap E\right]$$

$$= 0.$$

Therefore,

$$\nu[(T\times T)^i E\cap E] = \nu[(T\times T)^i(E'\sqcup E'')\cap E]$$

$$= \nu\left[\big((T\times T)^i E\sqcup (T\times T)^i E''\big)\cap E\right.$$

$$= \nu[(T\times T)^i E'\cap E\sqcup (T\times T)^i E''\cap E]$$

$$(3) \qquad \leq 0.$$

This holds for $1\leq i\leq h_{k+1}$. By induction, (3) is true for all $i\in\mathbb{N}$. There exists no integer i such that $\nu[(T\times T)^i E\cap E]>0$. Hence, $T\times T$ is not conservative.

□

2.4. **Zero Type.**

Definition 2.8. A measure preserving transformation T is of positive type if

$$\limsup \mu(A\cap T^{-n}(A))>0.$$

It is clear that a transformation of positive type is conservative.

Definition 2.9. A measure preserving transformation T is of zero type if

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(A)) = 0$$

for all $A \in \mathcal{S}$ with $\mu(A) < \infty$. Note that a transformation need be neither positive type nor zero type. However, in the case of *ergodic invertible* transformations in which rank one transformations satisfy, it is the case that they are either of positive or zero type. Readers may refer to [8] for proof.

Theorem 2.10. Let $(X, \mathfrak{B}(X), \mu, T)$ be an invertible conservative ergodic measure preserving transformation of positive type, then $\underbrace{T \times \cdots \times T}_{\text{d-times}}$ is of positive type (and hence conservative) for all $d \geq 1$. [1]

For the transformation in which we are considering, $T \times T$ is not conservative and hence, by Theorem 2.10, T is not of positive type. Since T is an ergodic invertible rank one transformation, we conclude that T must be of zero type.

2.5. **Power Cartesian Product of T.** In [3], it has been shown that there exists an infinite ergodic index transformation M that is of positive type and therefore all cartesian products are conservative. However, $M \times M^2$ is not conservative.

We have show earlier that T is of zero type. In this section, we show that with certain assumption of r_n, $T \times T^2$ is also not conservative.

Theorem 2.11. If $\sum_{n=1}^{\infty} \frac{1}{r_n} < \infty$, then the transformation $T^2 \times T$ is also non-conservative.

Proof. Since the series converges, we know that there exists an integer $n > 0$ such that

$$\sum_{k=n}^{\infty} \frac{1}{r_k} < \frac{1}{2}.$$

Let A be the top level of C_n. For $k \geq n$, let $C_{k,l}$ denote the l^{th} subcolumn of column C_k. Let $A_{k,l} = A \cap C_{k,l}$ denote copies of A in subcolumn $C_{k,l}$. Let $\nu = \mu \times \mu$ be the product measure. Define

$$E = (A \times A) \setminus \left(\bigcup_{k=n}^{\infty} \bigcup_{m=1}^{r_k} (A_{k,m} \times A_{k,m-1} \sqcup A_{k,m} \times A_{k,m}) \right).$$

$$\mu(E) \geq \mu(A)^2 \left(1 - \sum_{k=n}^{\infty} \frac{2}{r_k} \right)$$

Note that the measure of E is positive due to our choice of n. We will show by induction that for any integer $k \geq n$,

$$\nu((T^2 \times T)^i E \cap E) = 0 \tag{1}$$

for $1 \leq i \leq h_k$.

For the base case ($k = n$), since A is a top level in C_n and the spacers have the height at least h_n, the distance between $A_{n,l}$ and $A_{n,l+1}$ is $2h_n$ at most. If $1 \leq i < h_n$, then $2i < 2h_n$. Thus,

$$\nu[(T^2 \times T)^i E \cap E] \leq \nu[(T^2 \times T)^i A \times A \cap (A \times A)] = 0. \tag{2}$$

Next, suppose that $\nu((T \times T)^i E \cap E) = 0$ is true for $1 \leq i < h_k$, $k > n$. We will show that (1) is true for $h_k \leq i < h_{k+1}$.

Let l be an integer and $0 \leq l < r_k$. Note that place spacers of height $(2^{r_k - l} - 1)h_k$ on subcolumn $C_{k,l}$. We can see that for $h_k \leq i \leq (2^{r_k-l} - 1)h_k$, $\mu(T^i(A_{k,l})) \cap A_k = 0$.

In addition, the number of iterations from the bottom level of subcolumn $C_{k,l}$ to reach the first spacer on the last subcolumn (C_{k,r_k}) is $(2^{r_k-l+1} - 1)h_k$.

Thus, $\mu(T^i A_{k,l} \cap A_k) = 0$ for $(2^{r_k-l+1} - 1)h_k \leq i \leq H - h_k$ where H the number of spacers in the last subcolumn. We can extend this result up to $i \leq h_{k+1}$ as follow.

Consider the column C_{k+1}. For $(2^{r_k-l+1} - 1)h_k \leq i \leq H - h_k$, we have that $T^i(C_{k,l})$ is contained in the spacers on the subcolumn C_{k,r_k}. In C_{k+1}, the spacers become full levels, with additional spacers for column C_{k+1}, with height greater or equal to h_{k+1} on top. Therefore, if $(2^{r_k-l+1} - 1)h_k \leq i \leq h_{k+1}$, $T^i(C_{k,l})$ is either in the spacers of C_k or in those of C_{k+1}. This implies

$$\mu(T^i A_{k,l} \cap A_k) = 0$$

for all i such that $(2^{r_k-l+1} - 1)h_k \leq i \leq h_{k+1}$.

We conclude that if $\mu(T^i A_{k,l} \cap A_k) > 0$, then $i > (2^{r_k-l} - 1)h_k$ and $i < (2^{r_k-l+1} - 1)h_k$. Thus, define $I_l = \{h_k < i \leq h_{k+1} : \mu(T^i A_{k,l} \cap A_k) > 0\}$. We have

$$I_l \subset ((2^{r_k-l} - 1)h_k, (2^{r_k-l+1} - 1)h_k).$$

Denote $((2^{r_k-l} - 1)h_k, (2^{r_k-l+1} - 1)h_k)$ as I'_l. In addition, let $J_m = \{h_k < i \leq h_{k+1} : \mu(T^{2i} A_{k,m} \cap A_{k,m}) > 0.\}$. We have

$$J_{m-1} \subset (\frac{(2^{r_k-m+1} - 1}{2} h_k, \frac{2^{r_k-m+2} - 1}{2} h_k) = ((2^{r_k-m} - \frac{1}{2})h_k, (2^{r_k-m+1} - \frac{1}{2})h_k) \equiv J'_{m-1}$$

We observe that I_l overlaps J_{m-1} only when $l = m - 1$ and $l = m$. In addition, from definition, we have $\mu(T^{2i} A_{k,m} \cap A_{k,m}) > 0$ only when $i \in J'_{m-1}$ and $\mu(T^i A_{k,l} \cap A_{k,l}) > 0$ only when $i \in I'_l$. Thus, for $l \neq m, m-1$, either one of $\mu(T^{2i} A_{k,m} \cap A_{k,m})$ or $\mu(T^i A_{k,l} \cap A_{k,l})$ is zero.

Since

$$\nu[(T^2 \times T)^i (A_{k,m} \times A_{k,l}) \cap (A_k \times A_k)] = \mu[T^{2i} A_{k,m} \cap A_k] \cdot \mu[T^i A_{k,l} \cap A_k],$$

it follows that $\nu[(T^2 \times T)^i (A_{k,m} \times A_{k,l}) \cap (A_k \times A_k)] = 0$ for $l \neq m, m-1$.

Partition E into two parts as follow.

$$E = \left(E \cap \bigcup_{l \neq m, m-1} A_{k,m} \times A_{k,l} \right) \sqcup \left(E \cap \bigcup_{m=1}^{r_k} (A_{k,m} \times A_{k,m} \sqcup A_{k,m} \times A_{k,m-1}) \right) \equiv E' \sqcup E''$$

Observe that both $\nu[(T^2 \times T)^i E' \cap E] = 0$ and $\nu[(T^2 \times T)^i E'' \cap E] = 0$. Therefore,

$$\nu[(T^2 \times T)^i E \cap E] = 0$$

for for $1 \leq i \leq h_{k+1}$. By induction, (3) is true for all $i \in \mathbb{N}$. There exists no integer i such that $\nu[(T^2 \times T)^i E \cap E] > 0$. Hence, $T^2 \times T$ is not conservative.

□

3. Power Weak Mixing

We consider a family of cutting and stacking transformations such that the number of subcolumns formed to generate the next column is fixed.

Definition 3.1. A transformation T is called **power weakly mixing** if for any positive integer $r > 0$ and a given sequence of integers $(k_1, k_2, \ldots, k_r)$, the transformation $T^{k_1} \times \cdots T^{k_r}$ is ergodic. That is, for any sets A, B of positive measure, there exists an integer $m > 0$ such that

$$\mu(A \cap (T^{k_1} \times \cdots T^{k_r})^m(B)) > 0.$$

3.1. **Construction of a Power Weak Mixing Transformation.** We construct a class of transformations that can be shown to be power weakly mixing given that some conditions are met.

First, start with column $C_0 = [0, 1)$. To form a column C_{n+1} from C_n, cut the column into t subcolumns where $t \geq 3$, denoted by $C_{n,1}, \ldots, C_{n,t}$. We put a single spacer on the last subcolumn $C_{n,t}$. We may place h_n spacers(tower spacers) on top of some subcolumns; each construction can yield different mixing property.

Note that if there are q towers, each of which has h_n spacers, the number of subsections in C_{n+1} will be $t + q$. Let c denote $t + q$. If I is a level in column C_n, we will use the notation $I^{[k]} = I \cap C_{n,k}$ to denote the copy of I in the subcolumn $C_{n,k}$.

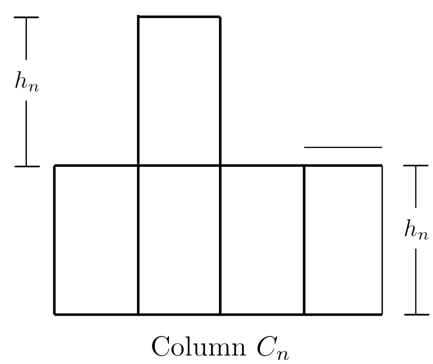

Figure 7. Construction of class of transformation T where $t = 4$ and a tower is placed every second subcolumn $C_{n,1}$. This transformation has been proved to be power weakly mixing in [5].

3.2. **Power Weakly Mixing Property.** Let T denote the class of transformation constructed above. We present conditions that imply power weakly mixing property of T.

Proposition 3.2. The transformation T with the following properties are power weakly mixing:

i.) $t \geq 3$.
ii.) Let I be a full level in column C_n. For $k \in \{0, 1, \ldots, c-1\}$, column C_{n+1} contains at least one full level as a copy of $T^{kh_n}(I)$.
iii.) For any $k \in \{1, \ldots, c-1\}$, $T^{kh_n}(I)$ contains at least one crescent structure that occupies every level below I in the given subcolumn with positive measure.

Let T be the transformation that satisfies all the properties in Proposition 3.2. Lemma 3.3 to Theorem 3.7 outlines the proof that T is power weakly mixing.

Lemma 3.3. Given measurable sets $A, B \subset \prod_{i=1}^{r} X$ of positive measure and $\epsilon > 0$, there exists rectangles $I = I_1 \times \ldots \times I_r$ and $J = J_1 \times \ldots \times J_r$ in a column C_k of transformation T such that for each $m \in \{1, \ldots, r\}$, I_m can be either above or below J_m and

$$\nu(I \cap A) > (1-\epsilon)\nu(I),$$

and

$$\nu(J \cap B) > (1-\epsilon)\nu(J).$$

Proof. Since a collection of rectangles form a sufficient semi-ring for the product space, given $\epsilon > 0$, we can choose a rectangle $I' = I'_1 \times \ldots \times I'_r$ such that I' is $(1 - \frac{\epsilon}{c^r})$-full of A. We may assume that I'_m are levels in the same column C_{k-1}. In addition, we can choose $J' = J'_1 \times \ldots J'_r$ such that it is $(1 - \frac{\epsilon}{c^r})$-full of B. We may assume that J'_m are levels in C_{k-1} as well.

Consider copies of C_{k-1} in the second and the fifth sections of C_k. To have I_m above J_m, let I_m be the copy of I'_m in the fifth section in C_k and let J_m be the second copy of J'_m in C_k. Let $I = I_1 \times \ldots I_r$ and $J = J_1 \times \ldots J_r$. It can be shown that I and J are $(1-\epsilon)$ full of A and B respectively.

□

Lemma 3.4. (Double Approximation Lemma) Suppose A is a subset of the product space $\prod_{i=1}^{r} X$ with $\nu(A) > 0$. Let $I = I_1 \times \ldots \times I_r$ be a rectangle in C_l that is $(1-\epsilon)-$ full of A. For $n > l$, the number of copies of C_l in C_n is $P_n = c^n$ where c is the number of subsections formed performed on C_i to obtain C_{i+1}. Let V_n index P_n copies of C_l in C_n, and let $V = V(n, r) = V_n \times \ldots V_n$ (r times).

Then for a given δ, $0 < \delta < 1$, and for any $\tau, 0 < \tau < 100(1-\epsilon)$, there exists an integer N such that for all $n > N$, there is a subset V'' of the index set V of size at least τ percent of V such that for all element $v = (v_1, \ldots, v_r) \in V''$, I_v is $(1-\delta)$ full of A and each I_v is of the form $I_v = I''_1 \times \ldots I''_r$ where I''_m is a sublevel of I_m in the v_m^{th} copy of C_l in C_n, $m \in \{1, \ldots, r\}$.

Proof. Redefine A as $A \cap I$. This is legitimate because we will only be interested in $I \cap A$.

Let $t = \tau/100$. Since I is $(1-\epsilon)$ full of A, $\mu(I \cap A) > (1-\epsilon)\mu(I)$. Equivalently, $\mu(I \setminus A) < \epsilon\mu(I)$. We have that $V_n = \{1, \ldots, P_n\}$ and $V = \{v_1, \ldots, v_r\} | v_i \in V_n\}$. Then, $I = \bigcup_{v \in V} I_v$.

Choose $c > \frac{\delta+1}{1-t-\epsilon}$. Pick $N > l$ so that for any $n \geq N$, there exists V' a subset of the index set V such that $I' = \bigcup_{v \in V'} I_v$ satisfies

$$\nu(I' \backslash A) < \frac{\delta}{c}\nu(I).$$

Thus,

$$\begin{aligned}
\nu(I' \backslash I) &\leq \nu(A \backslash I') + \nu(I \backslash A) \\
&< \frac{\delta}{c}\nu(I) + \epsilon\nu(I) \\
&= \left(\frac{\delta}{c} + \epsilon\right)\nu(I).
\end{aligned}$$

Let V'' be a set of indices v such that I_v is $(1-\delta)-$full of A. That is,

$$V'' = \{v \in V' \mid \nu(I_v \backslash) < \delta\nu(I_v)\}$$

and set $I'' = \bigcup_{v \in V''} I_v$, the union of the $(1-\delta)-$full subintervals. Then,

$$\begin{aligned}
\delta\nu(I' \backslash I'') &= \delta\nu\left(\bigsqcup_{v \in V' \backslash V''} I_v\right) \\
&= \sum_{v \in V' \backslash V''} \delta I_v \\
&\leq \sum_{v \in V' \backslash V''} \nu(I_v \backslash A) \\
&= \nu\left(\bigsqcup_{v \in V' \backslash V''} I_v \backslash A\right) \\
&\leq \nu(I' \backslash A)
\end{aligned}$$

To compare the size of V'' to V, we can consider the measure of I'' and I.

$$\begin{aligned}
\nu(I \backslash I'') &\leq \nu(I' \backslash I'') + \nu(I \backslash I') \\
&\leq \frac{1}{\delta}\nu(I \backslash A) + \nu(I \backslash I') \\
&< \frac{1}{c}\nu(I) + (\frac{\delta}{c} + \epsilon)\nu(I) \\
&< (1-t)\nu(I)
\end{aligned}$$

Hence, more then τ percent of subrectangles (I_v where $v \in V''$) are $(1-\delta)-$full of A.

□

Lemma 3.5. Let I, J be levels of a column C_n, $n > 0$, with J below I. Let d be the distance between I and J, that is, $T^d J = I$. If $k \in \{0, 1, \ldots, c-1\}$, then $T^{kh_n} I$ contains a copy of I at least in one section of C_{n+1} that is at distance d from a copy of J in C_{n+1}.

Furthermore, if $k \neq 0$,

$$\mu(T^{kh_n} I \cap J) > 0$$

Proof. The first part is already satisfied by the definition in Proposition 3.2. The second part is also an immediate result from 3.2*(iii)* because there is a copy of $T^{kh_n} I$ that occupies every level below the level of I.

Therefore, for $k \neq 0$,

$$\mu(T^{kh_n} I \cap J) > 0.$$

In fact, it can be shown that

$$\mu(T^{kh_n} I \cap J) = \frac{t - q - 1}{t^d} \mu(I) > 0.$$

□

Lemma 3.6. Let I and J be levels in column C_n, $n > 0$, where J is below I. Let d denote the distance and let k be any integer such that $0 < k < (c-1)d$. Then,

$$\mu(T^{kh_n} I \cap J) > \frac{1}{t^{d+k}} \mu(I).$$

Proof. Write k in base c as

$$k = \sum_{j=0}^{k'} k_j c^j$$

where $k_j \in \{0, 1, \ldots, c-1\}$ and $k' = \lfloor \log_c k \rfloor$.

Notice that $h_{n+1} = ch_n + 1$. Then,

$$\begin{aligned} c^j h_n &= c^j c^{-1} (h_{n+1} - 1) \\ &= c^{j-1} h_{n+1} - c^{j-1} \\ &= c^{j-2} h_{n+2} - \left(c^{j-1} + c^{j-2}\right) \\ &= h_{n+j} - \left(c^{j-1} + \ldots + c^1 + c^0\right) \\ &= h_{n+j} - \frac{1}{c-1} \left(c^j - 1\right) \end{aligned}$$

Thus,

$$kh_n = \sum_{j=0}^{k'} k_j c^j h_n = \sum_{j=0}^{k'} k_j h_{n+j} - \frac{1}{c-1} \sum_{j=0}^{k'} k_j (c^j - 1)$$

Note that

$$\frac{1}{c-1} \sum_{j=0}^{k'} k_j (c^j - 1) \leq \frac{1}{c-1} \sum_{j=0}^{k'} k_j c^j = \frac{k}{c-1} < d.$$

Let

$$a_0 = k_{k'} h_{n+k'},$$
$$a_1 = \sum_{j=0}^{k'-1} k_j h_{n+j}$$
$$a_2 = \sum_{j=2}^{k'} \frac{1}{c-1} k_j \left(c^j - 1\right)$$

Then,$T^{kh_n} = T^{(a_0+a_1-a_2)} I$. We first consider $T^{a_1} I$.

$$T^{a_1} I = T^{\sum_{j=0}^{k'-1} k_j h_{n+j}} I = T^{\sum_{j=1}^{k'-1} k_j h_{n+j}} \left(T^{k_0 h_n} I\right)$$

From the first part of Lemma 3.5, $T^{k_0 h_n} I$ is contains a full level in C_{n+1}. Apply Lemma 3.5 repeatedly, we conclude that $T_{a_1} I$ is contains a full level I' in $C_{n+k'}$.

By the second part of Lemma 3.5,

$$\frac{t-q-1}{t^d} \mu(I)$$

$$\begin{aligned}
\mu(T^{a_0}(T^{a_1} I) \cap J) &> \mu(T^{a_0}(I') \cap J) \\
&= \frac{t-q-1}{t^d} \mu(I') = \frac{t-q-1}{t^d} \cdot \frac{1}{t^{k'}} \mu(I) \\
&\geq \frac{1}{t^{d+k'}} \mu(I) \geq \frac{1}{t^{d+k}} \mu(I).
\end{aligned}$$

Finally, $T^{-a_2}(T^{a_0}(T^{a_1} I))$ moves the subcrescent down at most $\frac{k}{c-1} < d$ levels. Therefore, the subcrescent of I that has been translated down at most d levels still intersects the copy of J. The lower bound for the intersection still holds.

□

Theorem 3.7. The transformation T is power weakly mixing.

Proof. We'll show that for any $r \in \mathbb{Z}^+$ and any sequence of non-zero integers $\{k_1, \ldots, k_r\}$, the transformation $T_{k_1} \times \ldots \times T_{k_r}$ is ergodic.

Let $K = \max\{|k_i|\}$. Let $A, B \subset \prod_{i=1}^r$ be measurable sets with $\nu(A) > 0$ and $\nu(B) > 0$. By Lemma 3.3, choose rectangles $I = I_1 \times \ldots \times I_r$ and $J = J_1 \times \ldots \times J_r$ such that

$$\nu(A \cap I) > \frac{3}{4} \nu(I),$$
$$\nu(B \cap J) > \frac{3}{4} \nu(J),$$

and I_m, J_m where $m \in \{1, \ldots, r\}$ are all in the same column C_l, and if k_m is positive, choose I_m and J_m in the higher and lower sections of C_l

respectively, and if k_m is negative, choose I_m and J_m in the lower and the higher sections of C_l respectively.

We assume without loss of generality that $K < \frac{c-1}{c}h_l$. Let d_i be the distance between I_i and J_i for all i, and let $d = \max\{d_i\}$. Since $d > h_l/c$, it follows that $K < (c-1)d$. Choose δ so that

$$0 < \delta < \left(\frac{1}{t^{K+d}}\right)^r$$

By the Double Approximation Lemma, choose $I' = I'_1 \times \ldots \times I'_r$ such that I' are $(1 - \frac{\delta}{2})$ full of A. Suppose that I'_m are levels in column C_{n_I}. In addition, apply the Double Approximiation Lemma to obtain $J' = J'_1 \times \ldots \times J'_r$ such that J' are $1 - \frac{\delta}{2}$ full of B. Suppose J'_m are levels in column C_{n_I}. Let $n = \max\{n_I, n_J\}$. It follows that I' and J' where I'_m, J'_m are in column C_n are still at least $(1 - \frac{\delta}{2})$ full of A and B, respectively.

Since the Approximation Lemma guarantees that there are at least 75% of copies with the properties aforementioned, we can choose a common C_l−copy so that I'_m and J'_m are in the same column. Hence, the distance between I'_m and J'_m is still d_m. Let $H = h_n$.

For all positive k_m, by Lemma 3.6,

$$\mu(T^{k_i H} I'_m \cap J'_m) \geq \frac{1}{t^{k_i + d_i}} \mu(I'_m) \geq \frac{1}{t^{K+d}} \mu(I'_m)$$

Similarly, for all negative k_m, by Lemma 3.6,

$$\mu(T^{k_i H} I'_m \cap J'_m) = \mu(I'_m \cap T^{|k_i| H} J'_m) \geq \frac{1}{t^{k_i + d_i}} \mu(J'_m) \geq \frac{1}{t^{K+d}} \mu(I'_m)$$

Therefore,

$$\nu\left((T^{k_1} \times \ldots \times T^{k_r}) I' \cap J'\right) \geq \left(\frac{1}{t^{K+d}}\right)^r \nu(I')$$

Let $F = T^{k_1} \times \ldots \times T^{k_r}$. Relabel A to be $A \cap I$ and B to be $B \cap J$, we have

$$\begin{aligned} \nu\left(F^H A \cap B\right) &\geq \nu(F^H(I') \cap J') - \nu(F^H(I) \backslash A) - \nu(J \backslash B) \\ &\geq \left(\frac{1}{t^{K+d}}\right)^r \nu(J') - \frac{\delta}{2}\nu(I') - \frac{\delta}{2}\nu(J') > 0 \end{aligned}$$

due to our choice of δ. Hence, $T^{k_1} \times \ldots \times T^{k_r}$ is ergodic and T is power weakly mixing.

□

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